Sequential Games and the Extensive Form

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Introduction

By the end of this unit, students should be able to:

- Explain the concept of game trees and the extensive form,
- Define the various elements of games and game trees,
- Use best responses and backward induction to solve basic extensive form games, and
- Explain some of the contexts in which Nash and subgame perfect Nash equilibria are/aren't reasonable.

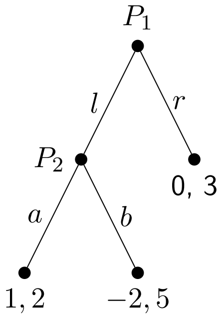
What goes in a game?

Every game will have the following elements:

- Players: Let $i \in N$ represent each player, i, in the population, N, such that $N = \{P_1, P_2, \ldots, P_n\}$.
- Actions: Let $a_i \in A_i$ represent a given action available to some player, i, out of all of their feasible actions, A_i.
- Preferences over payoffs: Let $u_i(x)$ denote a player's utility over payoff x and $u_i(y)$ represent their utility over payoff y. We will say i prefers x to y if $u_i(x) \geq u_i(y)$.

What is a game tree?

- A graphical representation that maps actions and payoffs over accumulated decisions into branches and nodes.
- In game theory, we refer to this layout as the "extensive form."

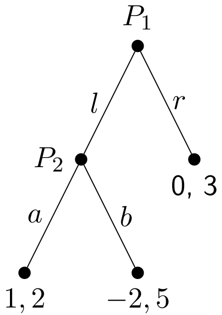


Game Tree 1

Nodes, branches, and paths of play

Game trees consist of:

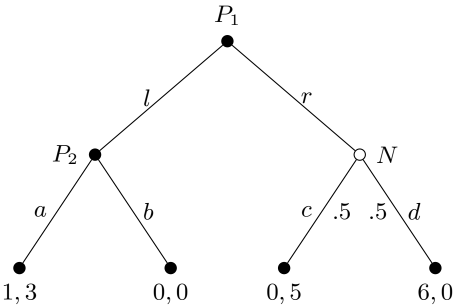
- Players: Labeled at the nodes
- Nodes: Root, decision, and terminal
- Branches: Indicate permissable actions
- Payoffs: Listed in order of play at terminal nodes



Game Tree 2

Extensive form games with chance nodes

- Sometimes we want to model a situation of uncertainty.
- We can imagine that Nature (N) becomes a player and chooses among its actions probabilistically.
- I'll use hollow nodes to denote moves by N.



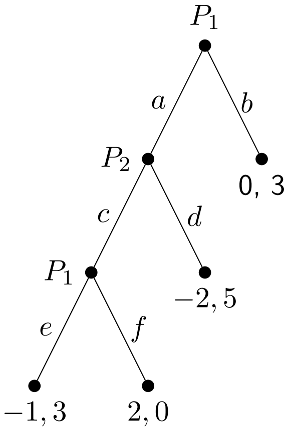
Game Tree 3

Strategy profiles

- Remember, a "strategy profile" is a *complete plan of action* for what decision one would make under any contingency.
- Let $s_i \in S_i$ represent a given strategy profile among all available to i.
- Next, let $\boldsymbol{s} = \{s_1, s_2, \ldots, s_n\}$ represent the individual strategy profiles selected by each player in a given game.
- Finally, let $\boldsymbol{S}$ represent all feasible combinations of strategy profiles players could submit.

Strategy profiles with game trees

- What is the set of strategy profiles available to each player in Game Tree 4?
- $S_1 = \{ae, af, be, bf\}$ and $S_2 = \{c, d\}$.
- Because P_1 has 4 feasible profiles, and P_2 has 2, there are 8 possible ways this game can be played.
- $\boldsymbol{S} = S_1 \times S_2$, or $\boldsymbol{S} = \{(ae, c); (ae, d); (af, c); (af, d); \ldots\}$.



Game Tree 4

Strategies, payoffs, and rationality

- Players have utility functions over the set of all feasible permutations such that payoffs are assessed according to the mapping function, $u_i(\boldsymbol{s}) : \boldsymbol{S} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.
- Rational choice theory holds that an individual will prefer to play a given strategy, s_i, over any other, s_i', so long as $u_i(s_i) \geq u_i(s_i')$.

The pure strategy Nash equilibrium (PSNE)

- A Nash equilibrium attains when no player may profitably deviate from a given strategy.
- More formally, a strategy profile, $\boldsymbol{s}^*$, is a Nash equilibrium when $u_i(s_i) \geq u_i(s_i', s_{-i}^*)$, for all i and s_i.

Deriving PSNE using best responses

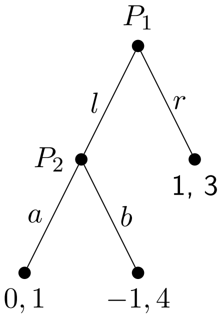
- One useful way to think about Nash equilibria is as a steady state of "best responses."
- A best response is a strategy profile that maximizes payoffs, holding competitors' strategies constant.
- When every player is playing strategies that are best-responding to one another, we are said to have a PSNE.
- Formally, a player, i can best best respond to the actions of other players, s_{-i} by selecting some:

$$B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \text{for all } s_i' \in S_i\}$$

- Therefore, a PSNE attains if $s_i^* \in B_i(s_{-i}^*)$ for all i.

Finding PSNE via best responses

- Consider this game.
  - If $s_1 = l$, then $B_2 = b$.
  - If $s_1 = r$, then $B_2 = a, b$.
  - If $s_2 = a$, then $B_1 = r$.
  - If $s_2 = b$, then $B_1 = r$.



Game Tree 5

Finding PSNE via best responses (cont'd.)

- If P_1 plays l, then P_2 best-responds with b. But because P_1 doesn't best respond with l to b, the profile, $\boldsymbol{s} = \{(l, b)\}$ is not a PSNE.
- If P_1 plays r, then P_2 best responds with either a or b. Because P_1 best responds with r in response to both a and b, then the strategy profiles $\boldsymbol{s} = \{(r, a)\}$ and $\boldsymbol{s} = \{(r, b)\}$ are each PSNE.
- Put differently, $PSNE = \{(r, a); (r, b)\}$.

Refining Nash in the extensive form

- In the previous slide, we found: $PSNE = \{(r,a);(r,b)\}$. But how realistic is the former of those two profiles?
- We found that P_2 was indifferent between her choices, but that was only because she was never actually called on to play.
- Had P_2 found herself called upon to make a decision, why would $s_2 = a$ be rational? Arguably, it would not be.

Subgame perfect Nash equilibria (SPNE)

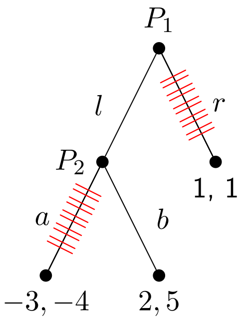
- The concept of "subgame perfection" refines the standard Nash equilibrium to require sequentially rational decision-making.
- That is, we should retain only those strategy profiles in equilibrium in which players make rational decisions within each "subgame."
- We can see from Game Tree 5 that at the decision node labeled, P_2, the only sequentially rational choice is $s_2 = b$. Therefore, we should reject the profile, $\boldsymbol{s}^* = \{(r, a)\}$, as it is not sequentially rational.
- Once we have done so, the set of subgame perfect Nash equilibria is $SPNE = \{(r, b)\}$.

Backward induction

- With extensive form games, finding SPNE is trivially simple using “backwards induction.”
- In a process we sometimes call, “pruning the game tree,” we start at terminal nodes and work our way backward to the root node, eliminating irrational decisions within each subgame.
- Let’s consider a few examples.

Pruning a simple game tree

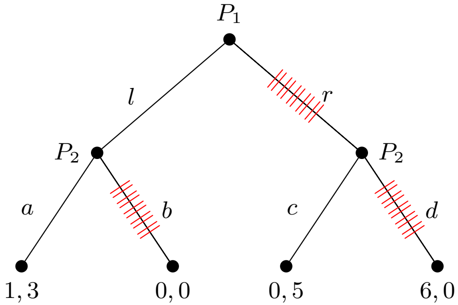
- There are 2 subgames in Game Tree 6. One begins at the node labeled P_2; the other is the game as a whole labeled at P_1.
- Starting at P_2, a is not a rational choice $(5 > -4)$, so we prune that branch.
- Moving to P_1, choosing r yields a payoff of 1 while l yields 2. Because $2 > 1$, P_1 chooses l and prunes r.
- Thus, the unique $SPNE = \{(l, b)\}$.



Game Tree 6

Pruning a more complicated game tree

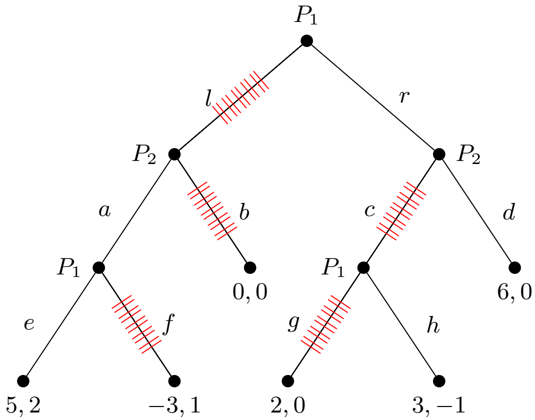
$SPNE = \{(l, ac)\}$



Game Tree 7

Pruning a more complicated game tree

$SPNE = \{(reh, ad)\}$



Game Tree 8

Order advantages

- Players could uniquely benefit in a sequential game simply due to the order in which they are called upon to play.
- Many games will favor the player who gets to move first, known as a "first-mover advantage" because such an advantage allows them to "set the agenda."
- For example, when the President nominates a Supreme Court justice (subject to Senate confirmation), the Senate can either accept or reject the nominee but can't choose for the President.
- In other situations, moving first can be a disadvantage as it allows other players to condition their decisions upon their predecessor's.

The market entry game

- Suppose an internet startuup is considering launching a video streaming app to compete directly with Netflix (we'll ignore other competitors).
- If the startup stays out of this market, it earns 0 profit while Netflix earns 1.
- If the startup enters the market, Netflix decides whether to accommodate this new competition or to engage in a (mutually destructive) price war.
- If Netflix accommodates, each firm earns 0.5. But if Netflix starts a price war, it earns 0 while the startup earns -1.
- What's the SPNE?

The market entry game (part ii)

- Building on the previous game, suppose we give the startup the opportunity to move again provided Netflix has chosen a price war.
- After Netflix sparks a price war, the startup can either choose to stay in the marketplace or leave.
- If it stays, each company continues to pay the costs of the war, and the startup earns -1 while Netflix earns 0.25.
- if the startup decides to leave, it cuts its losses and earns -0.5 while Netflix recuperates some of its market and earns 0.75.
- All other payoffs remain the same. What's the SPNE now?
- What's the SPNE?

Divide-the-dollar game (ultimatum game)

- Suppose George and John are part of a research study where the researcher gives George a dollar.
- George is instructed that he must offer John a portion of that dollar in one-cent intervals (0 cents to 100 cents).
- After George makes his offer, John can either accept his offer (in which case the payout occurs) or reject it (in which case each get nothing).
- What is the subgame perfect Nash equilibrium? How realistic is it?

The centipede game

- Two subjects are assigned the role as “first-mover” and “second-mover.”
- A researcher presents the subjects with a dollar. The first mover can pocket the dollar (such that the second-mover gets nothing) or pass it to the second mover.
- If the first-mover passes, the researcher adds another dollar to the pot, and the second-mover makes the same decision: pocket or pass.
- This sequence plays out until the pot is worth 5 dollars (whose move is that?)
- What’s the subgame perfect Nash equilibrium, and how realistic is this?

The brinksmanship game

- In 1962, the USSR installed nuclear missiles in Cuba.
- The US could heighten tensions by escalating the situation (air strike, trade embargo, etc.), or it could do nothing and allow the missiles to stay.
- In response to the US's decision, the USSR could either maintain its position or retreat.
- If the US escalates, and the USSR maintains, total war ensues, and players earn $(-1, -1)$, respectively.
- If the US escalates, and the USSR retreats: $(4, 2)$.
- If the US acquiesces and the USSR maintains: $(2, 4)$.
- And if the US acquiesces and the USSR retreats, peace prevails: $(3, 3)$.
- What's the SPNE?

Conclusion

- Sequential games allow us to model mutually strategic decisions while accounting for the kinds of real-life interactions players are likely to have.
- Nash and subgame perfect Nash equilibrium concepts furthermore allow us to think about what kinds of decision-making are rational.
- We have seen several examples where the solution concept produces realistic results and some that are less realistic.
- As we continue in this course, we will continue to refine our solution concept to arrive at increasingly realistic results.